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Mal'cev and Some Algebras

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ABSTRACT. In this paper we show that there is a 1-1 correspondence between Mal'cev algebra built on *C*-algebra and *C*-algebra built on Mal'cev algebra under certain conditions.

Introduction

In this paper, we build Mal'cev algebra on some other algebras, like *C*-algebra, Boolean algebra, Hyting algebra, and study some properties of this new form, and vice versa we build *C*-algebra on Mal'cev algebra and we find that there exists a 1-1 correspondence between Mal'cev algebra and *C*-algebra under certain conditions.

I. Mal'cev Algebra Built on *C*-Algebra

Definition 1

Mal'cev algebra is a set together with a ternary operation[1].

$$\theta : A \times A \times A \rightarrow A$$

which satisfies the following.

$$\theta(a, a, b) = b \qquad \theta(a, b, b) = a$$

and θ satisfies some properties as :

(1) θ is associative if

$$\theta(\theta(a, b, c), d, e) = \theta(a, b, \theta(c, d, e))$$

(2) θ is left distributive if

$$\theta(a, b, \theta(c, d, e)) = \theta(\theta(a, b, c), \theta(a, b, d), \theta(a, b, e))$$

(3) θ is right distributive if $\theta(\theta(a, b, c), d, e) = \theta(\theta(a, d, e), \theta(b, d, e), \theta(c, d, e))$

(4) θ is commutative if

$$\theta(a, b, c) = \theta(c, b, a)$$

(5) θ is abelian if

$$\theta(\theta(a, b, c), \theta(x, y, z), \theta(u, v, w)) = \theta(\theta(a, x, u), \theta(b, y, v), \theta(c, z, w))$$

(6) θ is middle distributive if

$$\theta(a, \theta(b, c, d), e) = \theta(\theta(a, d, e), \theta(a, c, e), \theta(a, b, e))$$

Definition 2

Comparison algebra (or C-algebra) is a set Q with a quaternary operation c which satisfies the following[2-4]

(c-1) $c(a, a, x, y) = x$

(c-2) $c(a, b, x, x) = x$

(c-3) $c(a, b, x, y) = c(b, a, x, y)$

(c-4) $c(a, b, a, b) = b$

(c-5) $c(a, b, c(x_i), c(y_i)) = c(c(a, b, x_i, y_i))$

where

$$c(x_i) = c(x_1, x_2, x_3, x_4) \text{ and } c(y_i) = c(y_1, y_2, y_3, y_4)$$

Theorem 1

Let Q be a C-algebra. If $\theta : Q \times Q \times Q \to Q$ such that $\theta(x, y, z) = c(x, y, z, x)$. The Q is a Mal'cev algebra

Proof

(1) $\theta(x, x, z) = c(x, x, z, x) = z$

(2) $\theta(x, y, y) = c(x, y, y, x) = c(y, x, y, x) = x$

Lemma 1

Let M be a Mal'cev algebra obtained from a C-algebra. Then $\theta(x, y, x) = x$,

Proof

$$\theta(x, y, x) = c(x, y, x, x) = x$$

II. Properties of Mal'cev Algebra Obtained from C-algebra

We show this in the following results :

2.1 Nonassociative

Proof

We need to show that

$$\theta(\theta(x, y, z), v, w) \neq \theta(x, y, \theta(z, v, w))$$

L.H.S. $= c(c(x, y, z, x), v, w, c(x, y, z, x))$.

R.H.S. $= c(x, y, c(z, v, w, z), x)$. If we take $y = w$, $z = v = x$, then

L.H.S. $= c(x, x, w, x) = w$

R.H.S. $= c(x, w, w, x) = x$.

Whenever $w \neq x$ the result follows.

2.2 Left distributive

Proof

We need to show that

$$\theta(a, b, \theta(c, d, e)) = \theta(\theta(a, b, c), \theta(a, b, d), \theta(a, b, e))$$

L.H.S. $= c(a, b, c(c, d, e, c), a)$

R.H.S. $= c(c(a, b, c, a), c(a, b, d, a), c(a, b, e, a), c(a, b, c, a))$

$= c(a, b, c(c, d, e, c), a)$.

2.3 It is not right distributive

Proof

We need to prove that

$$\theta(\theta(a, b, c), d, e) \neq \theta(\theta(a, d, e), \theta(b, d, e), \theta(c, d, e))$$

L.H.S. $= c(c(a, b, c, a), d, e, c(a, b, c, a))$

R.H.S. $= c(c(a, d, e, a), c(b, d, e, b), c(c, d, e, c), c(a, d, e, a))$

2.4 Not commutative

Proof

We need to prove that

$$\theta(a, b, c) \neq \theta(c, b, a)$$

L.H.S. $= c(a, b, c, a)$

R.H.S. $= c(c, b, a, c)$

As a special case, when $a \neq b \neq c$, $a = c$. L.H.S. $= a$ and R.H.S. $= a$. So, it is not abelian because it is not associative.

Definition 3

θ is said to be weak associative if

$$\theta(\theta(a, b, c), b, d) = \theta(a, b, \theta(a, b, d))$$

2.5 The Mal'cev algebra obtained from C-algebra is weak associative

Proof

$$\theta(\theta(a, b, c), b, d) = c(c(a, b, c, a), b, d, c(a, b, c, a))$$
$$\theta(a, b, \theta(c, b, d)) = c(a, b, c(c, b, d, c), a)$$

We have the following cases to consider :

Case (1). $a = b = c$

$$\theta(\theta(a, b, c), b, d) = c(c(a, a, a, a), a, d, c(a, a, a, a)) = c(a, a, d, a) = d$$

and

$$\theta(a, b, \theta(c, b, d)) = c(a, a, c(a, a, d, a), a) = c(a, a, d, a) = d$$

Case (2). $a = b, b, \neq c$

$$\theta(\theta(a, b, c), b, d) = c(c(b, b, c, b), a, d, c(b, b, c, b)) = c(c, b, d, c)$$
$$\theta(a, b, \theta(c, b, d)) = c(b, b, c(c, b, d, c), b) = c(c, b, d, c)$$

Definition 4

θ is called weak right distributive if

$$\theta(\theta(a, b, c,), b, e) = \theta(\theta(\mathrm{a, b, e}), e, \theta(c, b, e))$$

2.6 The Mal'cev algebra obtained from C-algebra is not weak right distributive

Proof

We need to show that

$$\theta(\theta(a, b, c), b, e) \neq \theta(\theta(a, b, e), e, \theta(\mathrm{c, b, e}))$$

L.H.S. $= c(c(a, b, c, a), b, e, c(a, b, c, a))$

R.H.S. $= c(c(a, b, e, a), e, c(c, b, e, c), c(a, b, e, a))$. Let $c \neq a \neq b \neq e, a = e$

L.H.S. $= c(a, b, e, a) = a$

R.H.S. $= c(a, e, c, a) = c$

III. C-algebra Built on Mal'cev Algebra

We will now build a C-algebra on a Mal'cev algebra by defining

$$c(a, b, x, y) = \theta(y, \theta(a, b, y), \theta(a, b, x)).$$

On a given set, we define θ_0 as follows :

$$\theta_0(a, b, x) = \begin{cases} x, & a = b \\ a, & a \neq b \end{cases}$$

Theorem 2

When $a = b$ and $x_i = y_i$ there is a correspondence between C-algebra and

Mal'cev algebra.

Proof

(i) at $a = b$

1. $c(a, a, x, y) = x$

1′. $\theta(y, \theta(a, a, y), \theta(a, a, x)) = \theta(y, y, x) = x$

2. $c(a, a, x, x) = x$

2′. $\theta(x, \theta(a, a, x), \theta(a, a, x)) = x$

3. $c(a, a, a, a) = a$

3′. $\theta(a, \theta(a, a, a), \theta(a, a, a) = a$

4. $c(a, a, x, y) = c(a, a, x, y)$, i.e., equal x

4′. $\theta(y, \theta(a, a, y), \theta(a, a, x)) = \theta(y, y, x) = x$

5. $c(a, a, c(x_i), c(y_i)) = c(c(a, a, x_i, y_i))$

5′. L.H.S. $= \theta[c(y_i), \theta(a, a, c(y_i)), \theta(a, a, c(x_i))] = \theta[c(y_i), c(y_i), c(x_i)]$
$= c(x_i) = c\ (x_1, x_2, x_3, x_4) = \theta[x_4, \theta(x_1, x_2, x_4), \theta(x_1, x_2, x_3)]$

R.H.S. $= c[\theta(y_i, \theta(a, a, y_i), \theta(a, a, x_i))] = c[\theta(y_i, y_i, x_i)] = c(x_i)$
$= c(x_1, x_2, x_3, x_4) = \theta[x_4, \theta(x_1, x_4, x_4), \theta(x_1, x_2, x_3)]$

(ii) at $x_i = y_i$, we need to prove that

$$c(a, b, c(x_i), c(x_i)) = c(c(a, b, x_i, x_i))$$

5′. L.H.S. $= \theta[c(x_i), \theta(a, b, c(x_i)), \theta(a, b, c(x_i))] = c(x_i) = c(x_1, x_2, x_3, x_4)$
$= \theta(x_4, \theta(x_1, x_2, x_4), \theta(x_1, x_2, x_3))$

R.H.S. $= c[\theta(x_i, \theta(a, b, x_i), \theta(a, b, x_i))],= c(x_i) = c(x_1, x_2, x_3, x_4)$
$= \theta[x_4, \theta(x_1, x_2, x_4), \theta(x_1, x_2, x_3)]$.

Thus, we prove that there is a 1-1 correspondence between C-algebra and Mal'cev algebra at (i) $a = b$ and (ii) $x_i = y_i$.

Lemma 2

If $a = b$, then any C-algebra built on a Mal'cev algebra satisfies the following properties:

(i) $c(a, b, b, a) = a$

(ii) $c(a, b, x, y) = c(a, b, x, c(a, b, x, y))$

(iii) $c(a, b, x, y) = c(a, b, c(a, b, x, y), y)$

Proof

(i) $c(a, b, b, a) = \theta(a, \theta(a, b, a), \theta(a, b, b)) = \theta(a, a, a) = a$

ii) L.H.S. $= \theta(y, \theta(a, b, y), \theta(a, b, x))$. At $a = b$, L.H.S. $= \theta(y, y, x) = x$

R.H.S. $= \theta[\theta[y, \theta(a, b, y), \theta(a, b, x)], \theta[(a, b, \theta(y, \theta(a, b, y), \theta(a, b, x))], \theta(a, b, x)]$. At $a = b$,

R.H.S. $= \theta[\theta(y, y, x), \theta[a, a, \theta(y, y, x)], \theta(a, a, x)] = \theta[x, x, x] = x.$

(iii) L.H.S. $= \theta(y, \theta(a, b, y), \theta(a, b, x))$.

R.H.S. $= \theta[y, \theta(a, b, y), \theta(a, b, \theta(y, \theta(a, b, y), \theta(a, b, x))]$.

At $a = b$,

R.H.S. $= \theta[y, y, \theta(y, \theta(a, a, y), \theta(a, a, x))] = \theta[y, y, \theta(y, y, x)] = \theta[y, y, x] = x$

L.H.S. $= \theta(y, y, x) = x$

Lemma 3

The following statements are equivalent:

(i) $\theta(s, \theta(a, b, s), \theta(a, b, x)) = \theta(s, \theta(a, b, s), \theta(a, b, y))$

(ii) $\theta(t, \theta(a, b, t), \theta(a, b, x)) = \theta(t, \theta(a, b, t), \theta(a, b, y))$

(iii) $y = \theta(y, \theta(a, b, y), \theta(a, b, x))$

Proof

(i) $\rightarrow$ (ii): Assume that $a = b$, then $\theta(s, \theta(a, a, s), \theta(a, a, x)) = \theta(s, \theta(a, a, s), \theta(a, a, y))$. Therefore, $\theta(s, s, x) = \theta(s, s, y)$ i.e. $x = y$. Therefore, by substituting in (ii)

L.H.S. $= \theta(t, \theta(a, a, t), \theta(a, a, x)) = \theta(t, t, x) = x$

R.H.S. $= \theta(t, t, y) = y$. By using condition (1), then L.H.S. = R.H.S.

(ii) $\rightarrow$ (iii): Assume then : $a = b$

R.H.S. $= \theta(y, \theta(a, a, y), \theta(a, a, x)) = \theta(y, y, x) = x$

L.H.S. $= y$

By using (ii), then $x = y$. (iii) $\rightarrow$ (i) similarly

Lemma 4

Any C-algebra built on Mal'cev algebra satisfies the following :

(i) There exists $s \in A$ such that

$$\theta(x, \theta(a, b, x), \theta(a, b, s)) = \theta(y, \theta(a, b, y), \theta(a, b, s))$$

(ii) $\theta(x, \theta(a, b, x), \theta(a, b, t)) = \theta(y, \theta(a, b, y), \theta(a, b, t))$

(iii) $x = \theta(y, \theta(a, b, y), \theta(a, b, x))$

Proof

At $a = b$

(i) $\theta(x, x, s) = \theta(y, y, s) = s$

(ii) $\theta(x, x, t) = \theta(y, y, t) = t$

(iii) $\theta(y, y, x) = x$

Definition 5

Given $a, b \in A$ define the relations $\psi(a, b)$ and $\Phi(a, b)$ as follow:

$x \psi y$ iff $y = \theta(y, \theta(a, b, y), \theta(a, b, x))$ and $x \Phi y$ iff $x = \theta(y, \theta(a, b, y), \theta(a, b, x))$
By using Lemma 3 we find that $x \psi y$ iff $x = y$. So ψ is an equivalence relation.

Lemma 5

Let a, b be given, then $\psi = \psi(a, b)$ and $\Phi = \Phi(a, b)$ and θ congruence relations.

Proof

If $x_i \psi y_i$ for $i = 1, 2, 3$ then $x_i = y_i$. Therefore, $\theta(x_i) = \theta(y_i)$. So $\theta(x_i) \psi \theta(y_i)$

Notation

A congruence relation t is trivial if it is the discrete congruence (i.e. xty only when $x = y$) indiscrete congruence (i.e., $xty \; \forall x, y$).

Definition 6

(A, θ) is called a simple θ-algebra if A has no trivial θ congruences.

Definition 7

$$\theta_0(y, \theta_0(a, b, y), \theta_0(a, b, x)) = \begin{cases} x & \text{if } a = b \\ y & \text{if } a \neq b \end{cases}$$

Theorem 3

If $a \neq b$. Then (A, θ) is a simple θ-algebra iff $\theta = \theta_0$.

Proof

If $\theta = \theta_0$ and if t is a congruence relation then $\forall (x, y)$ we have

$\theta(y, \theta(a, b, y), \theta(a, b, x)) \, t\theta_0(y, \theta_0(a, b, y), \theta_0(a, b, x)) \, t\theta_0(y, \theta_0(a, a, y), \theta_0(a, a, x))$

$\theta_0(y, \theta_0(a, b, y), \theta_0(a, b, x)) \, t\theta_0(y, \theta_0(a, a, y), \theta_0(a, a, x))$

From definition of θ_0, so xty. Conversely, if (A, θ) is simple and $\theta(y, \theta(a, b, y), \theta(a, b, x)) \neq \theta_0(y, \theta_0(a, b, y), \theta_0(a, b, x))$. Then let $\psi = \psi(a, b)$. Thus $a \psi b$ and ψ must be the indiscrete congruence. Therefore, $x \psi y$ so $y = \theta(y, \theta(a, b, y), \theta(a, b, x))$ and at $a \neq b$, $y = \theta_0(y, \theta_0(a, b, y), \theta_0(a, b, x))$ i.e., there is contradiction. Then, $\theta = \theta_0$.

IV. Mal'cev Algebra Structure on Boolen Algebra

Using a Boolean algebra $(A, \wedge, \vee, \sim)$, we define a Mal'cev algebra structure as follows[5-7]:

$$\theta(x, y, z) = (x \vee \sim y) \wedge (z \vee \sim y) \wedge (x \vee z)$$

4.1 It is no associative

Proof

We need to prove that

$$\theta(a, b, \theta(c, d, e)) \neq \theta(\theta(a, b, c), d, e)$$

Let 0 be false and 1 be true. When $a = 1$,

L.H.S. $= \theta(1, b, \theta(c, d, e)) = [\sim b \vee \theta(c, d, e)] = \sim b \vee [(c \vee \sim d) \wedge (e \vee \sim d) \wedge (c, \vee e)]$

R.H.S. $= \theta[\theta(1, b, c), d, e] = [\theta(1, b, c) \vee \sim d] \wedge (e \vee \sim d) \wedge [\theta(1, b, c) \vee e]$

$= [(\sim b \vee c) \vee \sim d] \wedge (e \vee \sim d) \wedge [(\sim b \vee c) \vee e]$

$= [\sim b \vee (c \vee \sim d)] \wedge (e \vee \sim d) \wedge [\sim b \vee (c \vee e)]$.

When $a = 0$,

L.H.S. $= \theta(0, b, \theta\, \theta(c, d, e)) = \sim b \wedge \theta(c, d, e) = b \wedge [(c \vee \sim d) \wedge (e \vee \sim d) \wedge (c \vee e)]$

R.H.S. $= \theta(\theta(0, b, c), d, e) = [\theta(0, b, c) \vee \sim d] \wedge [e \vee \sim d] \wedge [(\theta(0, b, c) \vee e]$

$= [(\sim b \wedge c) \vee \sim d] \wedge [e \vee \sim d] \wedge [(\sim b \wedge c) \vee e]$

$= [\sim b \wedge (c \vee \sim d)] \wedge [e \vee \sim d] \wedge [\sim b \wedge (c \vee e)]$

4.2 It is weak associative

Proof

Fix e in the algebra. Then

$$\theta(x, e, y) = [(x \wedge y) \vee \sim e] \wedge (x \vee y).$$

Therefore, $\theta(\theta(x, e, y), e, z) = \theta(x, e\, \theta(y, e, z))$.

4.3 It is left distributive

Proof

When $a = 1$, we want to prove that

$$\theta(1, b, \theta(x, y, z)) = \theta(\theta(1, b, x), \theta(1, b, y), \theta(1, b, z))$$

We note that $\theta(1, b, c) = \sim b \vee c$. So

L.H.S. $= \sim b \vee [(x \vee \sim y) \wedge (z \vee \sim y) \wedge (x \vee z)]$.

R.H.S. $= \theta[(\sim b \vee x), (\sim b \vee y), (\sim b \vee z)]$

$= [(\sim b \vee x) \vee \sim(\sim b \vee y)] \wedge [(\sim b \vee z) \vee \sim(\sim b \vee y)] \wedge [(\sim b \vee x) \vee (\sim b \vee z)]$

$= [(\sim b \vee x) \vee (b \wedge \sim y)] \wedge [(\sim b \vee z) \vee (b \wedge \sim y)] \wedge [(\sim b \vee x) \vee (\sim b \vee z)]$

$$= [((\sim b \vee x) \vee b) \wedge ((\sim b \vee x) \vee \sim y)] \wedge [(\sim b \vee z) \wedge b) \wedge ((\sim b \vee z) \vee \sim y)] \wedge [\sim b \vee (x \vee z)]$$

$$= [1 \wedge ((\sim b \vee x) \vee \sim y)] \wedge [1 \wedge ((\sim b \vee z) \vee \sim y)] \wedge [\sim b \vee (x \vee z)]$$

$$= [(\sim b \vee x) \vee \sim y] \wedge [(\sim b \vee z) \vee \sim y] \wedge [\sim b \vee (x \vee z)]$$

$$= \sim b \vee [(x \vee \sim y) \wedge (z \vee \sim y) \wedge (x \vee z)]$$

When $a = 0$, $\theta(0, b, c) = (\sim b \wedge c)$. We want to proof that

$$\theta(0, b, \theta(x, y, z)) = \theta[\theta(0, b, x), \theta(0, b, y), \theta(0, b, z)]:$$

L.H.S. $= \theta(0, b, \theta(x, y, z)) = \theta(x, y, z) \wedge \sim b$

$= [(x \vee \sim y) \wedge (z \vee \sim y) \wedge (x \vee z)] \wedge \sim b.$

R.H.S. $= \theta(x \wedge \sim b, y \wedge \sim b, z \wedge \sim b)$

$= [(x \wedge \sim b) \vee (\sim y \vee b)] \wedge [(z \wedge \sim b) \vee (\sim y \vee b)] \wedge [(x \wedge \sim b) \vee (z \wedge \sim b)]$

$= [(x \vee (\sim y \vee b)) \wedge (\sim b \vee (\sim y \vee b))] \wedge [(z \vee (\sim y \vee b)) \wedge (\sim b \vee (\sim y \vee b))] \wedge [(x \vee z) \wedge \sim b]$

$= [(x \vee (\sim y \vee b)) \wedge 1] \wedge [(z \vee (\sim y \vee b)) \wedge 1] \wedge [(x \vee z) \wedge \sim b]$

$= [x \vee (\sim y \vee b)] \wedge [(z \vee (\sim y \vee b)] \wedge [(x \vee z) \wedge \sim b]$

$= [[(x \vee \sim y) \wedge (z \vee \sim y)] \vee b] \wedge [(x \vee z) \wedge \sim b]$

Since $b \wedge (x \vee z) \wedge \sim b = 0$, so

R.H.S. $= [(x \vee \sim y) \wedge (z \vee \sim y) \wedge (x \vee z)] \wedge \sim b$

4.4 It is right distributive

Proof

We have $\theta(x, y, 1) = x \vee \sim y$.

We want to prove that $\theta(\theta(x, y, z), d, e) = \theta(\theta(x, d, e), \theta(\mathrm{y, d, e}), \theta(z, d, e))$

Let $e = 1$, then

L.H.S. $= \theta(\theta(x, y, z), d, 1) = \theta(x, y, z) \vee \sim d$

$= [(x \vee \sim y) \wedge (z \vee \sim y) \wedge (x \vee z)] \vee \sim d.$

R.H.S. $= \theta[\theta(x, d, 1), \theta(y, d, 1), \theta(z, d, 1)]$

$= \theta[(x \vee \sim d), (y \vee \sim d), (z \vee \sim d)]$

$= [(x \vee \sim d) \vee (\sim y \wedge d)] \wedge [(z \vee \sim d) \vee (\sim y \wedge d)] \wedge [(x \vee z) \vee \sim d$

$= [(x \vee \sim d) \vee \sim y] \wedge [(x \vee \sim d) \vee d] \wedge [(z \vee \sim d) \vee \sim y] \wedge [(z \vee \sim d) \vee d] \wedge [(x \vee z) \vee \sim d]$

$= [(x \vee \sim y) \vee \sim d] \wedge [(z \vee \sim y) \vee \sim d) \wedge [(x \vee z) \vee \sim d]$

$= [(x \vee \sim y) \wedge (z \vee \sim y) \wedge (x \vee z)] \vee \sim d$

Then $z = 0$. So $\theta(x, y, 0) = x \wedge \sim y$.

$$\begin{aligned}
\text{L.H.S.} &= \theta[\theta(x, y, z)\, d, 0] \\
&= \theta(x, y, z) \wedge \sim d = [(x \vee \sim y) \wedge (z \vee \sim y) \wedge (x \vee z)] \wedge \sim d. \\
\text{R.H.S.} &= \theta[\theta(x, d, 0), \theta(y, d, 0), \theta(z, d, 0)] \\
&= \theta[x \wedge \sim d, y \wedge \sim d, z \wedge \sim d] \\
&= [(x \wedge \sim d) \vee \sim (y \wedge \sim d)] \wedge [(z \wedge \sim d)] \vee \sim (y \wedge \sim d)] \\
&\quad \wedge [(z \wedge \sim d) \vee (x \wedge \sim d)] \\
&= [(x \wedge \sim d) \vee (\sim y \vee d)] \wedge [(z \wedge \sim d)] \vee (\sim y \vee d)] \wedge [(z \wedge \sim d) \\
&\quad \vee (x \wedge \sim d)] \\
&= [(x \vee \sim y) \vee d] \wedge [\sim d \vee (\sim y \vee d)] \wedge [(z \vee \sim y) \vee d) \\
&\quad \wedge (\sim d \vee \sim y) \vee d] \wedge [(z \vee x) \wedge \sim d] \\
&= [(x \vee \sim y) \vee d] \wedge [(z \vee \sim y) \vee d)] \wedge [(z \vee x) \wedge \sim d]. \\
&= [[(x \vee \sim y) \wedge (z \vee \sim y)] \vee d] \wedge [(z \vee x) \wedge \sim d].
\end{aligned}$$

Using distributive law:

$$\begin{aligned}
\text{R.H.S.} &= [(x \vee \sim y) \wedge (z \vee \sim y) \wedge (z \vee x) \wedge \sim d] \wedge [d \wedge (z \vee x) \wedge \sim d] \\
&= [(x \vee \sim y) \wedge (z \vee \sim y) \wedge (z \vee x) \wedge \sim d].
\end{aligned}$$

4.5 It is middle distributive

Proof

To prove $\theta(a, \theta(b, c, d), e) = \theta(\theta(a, d, e), \theta(a, c, e), \theta(a, b, e))$, where $e = 1, 0$.

(i) Let $e = 1$. Then

$$\begin{aligned}
\text{L.H.S.} &= \theta(a, \theta(b, c, d), 1) = a \vee \sim \theta(b, c, d) = a \vee \sim [(b \vee \sim c) \wedge (d \vee \sim c) \\
&\quad \wedge (b \vee d)] \\
&= a \vee [(\sim b \wedge c) \vee (\sim d \wedge c) \vee (\sim b \wedge \sim d)]. \\
\text{R.H.S.} &= \theta(a \vee \sim d, a \vee \sim c, a \vee \sim b) \\
&= [(a \vee \sim d) \vee (\sim a \wedge c)] \wedge [(a \vee \sim b) \vee (\sim a \wedge c)] \wedge [(a \vee \sim b) \\
&\quad \vee (a \vee \sim d)] \\
&= [(a \vee \sim d) \vee \sim a] \wedge [a \vee \sim d) \vee c] \wedge [(a \vee \sim b) \vee \sim a] \\
&\quad \wedge [(a \vee \sim b) \vee c] \wedge [(a \vee (\sim b \vee \sim d)] \\
&= [(a \vee \sim d) \vee c] \wedge [(a \vee \sim b) \vee c] \wedge [a \vee (\sim b \vee \sim d)] \\
&= [a \vee (\sim d \vee c)] \wedge [a \vee (\sim b \vee c)] \wedge [a \vee (\sim b \vee \sim d)], \\
&= a \vee [(\sim d \vee c) \wedge (\sim b \vee c) \wedge (\sim b \vee \sim d)],
\end{aligned}$$

since

$a \vee [(\sim b \wedge c) \vee (\sim d \wedge c) \vee (\sim b \wedge \sim d)] = a \vee [(\sim b \vee c) \wedge (\sim d \vee c) \wedge (\sim b \vee \sim d)]$.

(ii) Let $e = 0$. Then

$$\begin{aligned} \text{R.H.S.} &= \theta(a, \theta(b, c, d), 0) = a \wedge \sim \theta(b, c, d) \\ &= a \wedge [(\sim b \wedge c) \vee (\sim d \wedge c) \vee (\sim b \wedge \sim d)] \\ \text{L.H.S.} &= \theta(\theta(a, d, 0), \theta(a, c, 0), \theta(a, b, 0)) \\ &= \theta[(a \wedge \sim d), (a \wedge \sim c), (a \wedge \sim b)]. \end{aligned}$$

By similar calculations as in the case $e = 1$ we find

$$\text{L.H.S.} = a \wedge [(\sim b \vee c) \wedge (\sim d \vee c) \vee (\sim b \vee \sim d)].$$

V. Mal'cev Algebras Based on Heyting Algebras

We define a Mal'cev algebra structure on a Heyting[8, 9] algebra as follows:

$$\theta(x, y, z) = ((x \to y) \to z) \wedge ((z \to y) \to x).$$

Example 1

The Heyting algebra of three elements $[0, a, 1]$,

$$a \to 0 = a,$$

$$a = \text{largest element } x \text{ such that } x \wedge 0 = 0,$$

5.1. It is commutative since $\theta(a, b, c) = \theta(c, b, a)$

5.2. It is not associative

Proof

We need to prove that

$$\theta(\theta(a, b, c), d, e) \neq \theta(a, b, \theta(c, d, e))$$

Let $a = c = d = 0$, $b = e$. Then

$$\begin{aligned} \text{L.H.S.} &= \theta(\theta((0, e, 0), 0, a) = \theta(0, 0, e) = e. \\ \text{R.H.S.} &= \theta(0, e, \theta(0, 0, e)) = \theta(0, e, e) = 0. \end{aligned}$$

5.3. It is not left distributive

Proof

We need to prove that

$$\theta(a, b, \theta(c, d, e)) \neq \theta(\theta(a, b, c), \theta(a, b, d), \theta(a, b, e))$$

Let $c = e = a$, $b = 1$, $d = 0$. Then

$$\text{L.H.S.} = \theta(a, 1, \theta(a, 0, a)) = \theta(a, 1, 1) = a.$$

R.H.S. $= \theta(\theta(a, 1, a), \theta(a, 1, 0), \theta(a, 1, a)) = \theta(a, 0, a) = 1.$

5.4. It is not right distributive

Proof

We need to prove that

$$\theta(\theta(a, b, c), d, e) \neq \theta(\theta(a, d, e), \theta(b, d, e), \theta(c, d, e))$$

Let $a = e = c$, $b = 0$, $d = 1$. Then

L.H.S. $= \theta(\theta(a, 0, a), 1, a) = \theta(1, 1, a) = a$

R.H.S. $= \theta(\theta(a, 1, a), \theta(0, 1, a), \theta(a, 1, a)) = \theta(a, 0, a) = 1.$

Remark

We note that any Mal'cev algebra built on a Heyting algebra is commutative, i.e.

$$\theta(x, y, z) = \theta(z, y, x).$$

Proof

$\theta(x, y, z) = ((x \rightarrow y) \rightarrow z) \wedge ((z \rightarrow y) \rightarrow x).$

$\theta(z, y, x) = ((z \rightarrow y) \rightarrow x) \wedge ((x \rightarrow y) \rightarrow z).$

Which are equal since $\wedge$ is commutative.

Example 2

Consider the Boolean algebra for the interval [0, 1]:

$(x \wedge y) = \min(x, y)$,

$(x \vee y) = \max(x, y)$,

$\sim x = 0$ if $x \neq 0$,

$\sim x = 1$ if $x = 0$.

So Heyting algebra on the interval [0, 1] is given by:

$$x \rightarrow y = \begin{cases} 1 & \text{if } x = 0 \\ y & \text{if } y \leq x \end{cases}$$

Example 3

Consider the Hyting algebra on [0, 1] that is given by :

$$x \rightarrow 1 = 1, 0 \rightarrow x = 1, x \rightarrow x = 1 \text{ and if } x \leq y \text{ then } x \rightarrow y = 1.$$

So it is not Boolean.

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مـــالسيف وبعـــض الجــــبريات

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المستخلص . في هذا البحث سنوضح أن هناك تناظرًا أحاديًا بين جبر مالسيف المنشأ على جبر C وبين جبر C المنشأ على جبر مالسيف وذلك تحت شروط معينة .