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Series Inversion of the Universal Kepler Equation

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ABSTRACT. In this paper, symbolic expression for the solution of the universal Kepler equation is developed using series reversion algorithm.

1. Introduction

During space mission of all types of two-body motion (elliptic, parabolic, or hyperbolic) appear. For example, the escape from the departure planet and the capture by the target planet, involve hyperbolic orbits, while the intermediate stage of the mission commonly depicted as a heliocentric ellipse, may also be heliocentric parabola or hyperbola. In addition, in some systems, the type of an orbit is occasionally changed by perturbing forces acting during finite interval of time. Thus far we have been obliged to use different functional representations for the motion depending on its type and a simulation code must then contain branching to handle a switch from one type to another. In cases where this switching is not smooth, branching can occur many times during a single integration time-step, causing some numerical "chatter". Consequently, universal formulations are desperately needed to that orbit predictions will be free of the troubles, since a single functional representation suffices to describe all possible orbits. As a result of this universalization, is the universal Kepler's equation which is highly transcendental and usually solved by iterative methods [Sharaf and Sharaf, 1998].

Undoubtedly true that, the numerical methods provide very accurate solutions. But certainly, if full analytical formulae are utilized with nowadays existing symbols used for manipulating digital computer programs, they definitely become invaluable for obtaining solutions with any desired accuracy. Moreover, symbolic computing algorithms for space dynamics problems represent new branch that may be called the algorithmization of space dynamics [Bumberg, 1995].

Coping with this line of recent researches and also due to the important role of the universal Kepler's equation, the present paper is devoted to develop symbolic expression for the solution of the universal Kepler equation.

2. Formulation

2.1 Series Reversion Algorithm

General algorithm for reversing a power series which is the fundamental technique of the subsequent analysis, will be developed as follows: Consider the functional equation

$$\eta = \xi + \alpha\,\phi\,(\eta)\,. \qquad |a| < 1\,, \tag{2.1}$$

then according to Lagrange expansion theorem (Smart, 1953), we have

$$\eta = \xi + \sum_{n=1}^{\infty} \frac{\alpha^n}{n!}\,\frac{d^{n-1}}{d\xi^{n-1}}\,[\phi(\xi)]^n\ . \tag{2.2}$$

Let $y(x)$ be a function which can be expanded in a Taylor series in the neighborhood of $x = x_0$. Thus

$$y(x) = y_0 + \sum_{j=1}^{\infty} \frac{B_j}{j!}\,(x - x_0)^j\ , \tag{2.3}$$

where

$$B_j = \left.\frac{d^j y(x)}{dx^j}\right|_{x=x_0}\ .$$

In the following, we assume that B_1 is different from zero and write Equation (2.3) in the form

$$x = x_0 + (y - y_0)\,\phi\,(x)\,, \tag{2.4}$$

where $\phi(x)$ is defined by

$$\phi(x)\ =\ \frac{1}{B_1 + \sum_{j=1}^{\infty}[B_{j+1}/(j+1)!]\ (x - x_0)^j}\ . \tag{2.5}$$

Equation (2.4) is precisely the same form as Equation (2.1), then we can express x as a power series in $\alpha = y - y_0$ to get

$$x(y) = x_0 + \sum_{n=1}^{\infty} \frac{C_n}{n!} (y - y_0)^n \ , \tag{2.6}$$

where

$$C_n = \left. \frac{d^{n-1}}{dx^{n-1}} [\phi(x)]^n \right|_{x=x_0} \tag{2.7}$$

and $\phi(x)$ is defined in Equation (2.5). The series for $x(y)$ is said to be the *reverse* of the series for $y(x)$.

Battin (1999) developed an elegent algorithm to express n of the coefficients C_1, C_2, ... of the reversed series in terms of the coefficients B_1, B_2, ... of the original series. The basic equations of this algorithm are

$$\left. D_0^1 \right|_{x=x_0} = \phi^{(0)}(x_0) \equiv \phi_0^{(0)} = \frac{1}{B_1} \ , \tag{2.8.1}$$

$$\frac{d^k \phi(x)}{dx^k} \equiv \phi_0^{(k)} = -\frac{1}{B_1} \sum_{i=1}^{k} \frac{1}{i+1} \binom{k}{i} B_{i+1} \phi_0^{(k-i)} \ , \quad k = 1, 2, \ \ldots \ h-1, \tag{2.8.2}$$

$$D_k^n = \frac{d^k}{dx^k} [\varphi(x)]^n = n \sum_{j=0}^{k-1} \binom{k-1}{j} D_j^{n-1} \phi_0^{(k-j)} \ , \tag{2.8.3}$$

$$C_n = \left. D_{n-1}^n \right|_{x=x_0} \ . \tag{2.8.4}$$

2.2 Universal Kepler's Equation

In the universal function representation, Kepler's equation relating time and position (the universal anomaly χ) is given by

$$\chi = \alpha \sqrt{\mu} \, (t - t_0) + \sigma - \sigma_0 \ , \tag{2.9}$$

where α is the reciprocal of the semimajor axis, μ is the gravitational parameter (universal gravitational constant times the mass of the central attracting body), t is the time and $\sigma = \boldsymbol{r}.\boldsymbol{v} / \sqrt{\mu}$, with $\boldsymbol{r}$ and $\boldsymbol{v}$ the position and velocity vectors, respectively, of the orbiting body; subscript 0 will denote evolution at the time t_0.

In Equation (2.9), the quantity σ can be replaced with an expression involving only χ as the variable (Battin, 1999),

$$\sigma = \sigma_0 \, U_0 \, (\chi \, ; \, \alpha) + (1 - \alpha \, r_0) \, U_1 \, (\chi \, ; \, \alpha), \tag{2.10}$$

where the $U_n \, (x \, ; \, \alpha)$ are the universal functions; defined by

$$U_n(\chi\ ;\ \alpha) = \chi^n \sum_{i=0}^{\infty} (-1)^i \frac{(\alpha\ \chi^2)^i}{(n+2i)!} \tag{2.11}$$

3. Symbolic Representation of χ

From Equations (2.10) and (2.11) into Equation (2.9) we get a series of the form

$$t - t_0 = A_1\ \chi + \frac{A_2}{2!}\chi^2 + \frac{A_3}{3!}\chi^3 + \ \ldots \tag{3.1}$$

(note that $A_0 = 0$ because $\chi = 0$ at $t = t_0$). Literal analytical expressions of A_j ; $j = 1, 2, \ldots, 20$ are listed in Table 1.

TABLE 1. Literal analytical expressions of A_j , $j = 1, 2, \ldots, 20$ of equation (3.1).

$A_1 = \frac{r_0}{\sqrt{\mu}}$	$A_2 = \frac{\sigma_0}{\sqrt{\mu}}$	$A_3 = \frac{1-\alpha\, r_0}{\sqrt{\mu}}$	$A_4 = -\frac{\alpha\, \sigma_0}{\sqrt{\mu}}$
$A_5 = \frac{\alpha\,(\alpha\ r_0 - 1)}{\sqrt{\mu}}$	$A_6 = \frac{\alpha^2\ r_0}{\sqrt{\mu}}$	$A_7 = \frac{\alpha^2 - \alpha^3\ r_0}{\sqrt{\mu}}$	$A_8 = -\frac{\alpha^3\, \sigma_0}{\sqrt{\mu}}$
$A_9 = \frac{\alpha^3\ (\alpha\ r_0 - 1)}{\sqrt{\mu}}$	$A_{10} = \frac{\alpha^4\ \sigma_0}{\sqrt{\mu}}$	$A_{11} = \frac{\alpha^4 - \alpha^5\ r_0}{\sqrt{\mu}}$	$A_{12} = -\frac{\alpha^5\ \sigma_0}{\sqrt{\mu}}$
$A_{13} = \frac{\alpha^5\ (\alpha\ r_0 - 1)}{\sqrt{\mu}}$	$A_{14} = \frac{\alpha^6\ \sigma_0}{\sqrt{\mu}}$	$A_{15} = \frac{\alpha^6 - \alpha^7\ r_0}{\sqrt{\mu}}$	$A_{16} = -\frac{\alpha^7\ \sigma_0}{\sqrt{\mu}}$
$A_{17} = \frac{\alpha^7\ (\alpha\ r_0 - 1)}{\sqrt{\mu}}$	$A_{18} = \frac{\alpha^8\ \sigma_0}{\sqrt{\mu}}$	$A_{19} = \frac{\alpha^8\ (1 - \alpha\ r_0)}{\sqrt{\mu}}$	$A_{20} = -\frac{\alpha^9\ \sigma_0}{\sqrt{\mu}}$

Reversing series (3.1) using the algorithm of Subsection 2.1 leads to a solution for $\chi(t, \alpha, s_0, \mathrm{r}_0)$,

$$\chi\ =\ \sum_{k=1}^{N} \frac{C_k}{k!}\ (t - t_0)^k\ , \tag{3.2}$$

where N is the order of the truncated series. The procedure was mechanized using software package Mathematica to generate the C's coefficients in terms of α, σ_0, r_0 and μ. Because of space limitations, only the first ten C's coefficients are listed in Table 2.

Conclusion

In concluding the present paper, symbolic expression for the solution of the universal Kepler's equation is developed using series reversion algorithm. This solution is useful analytic approximation that takes advantage of the singularity-free (as $e \rightarrow$) universal function formulation.

TABLE 2. Literal analytical expressions of C_j, j = 1, 2, ... , 10 of equation (3.2).

$$C_1 = \frac{\sqrt{\mu}}{r_0}$$

$$C_2 = -\frac{\mu\ \sigma_0}{r_0^3}$$

$$C_3 = \frac{\mu^{3/2}(-r_0 + \alpha\ r_0^2 + 3\ \sigma_0^2)}{r_0^5}$$

$$C_4 = \frac{\mu^2\ \sigma_0\ (10\ r_0 - 9\ \alpha\ r_0^2 - 15\ \sigma_0^2)}{r_0^7}$$

$$C_5 = \frac{\mu^{5/2}(-19\ \alpha\ r_0^3 + 9\ \alpha^2\ r_0^4 - 105\ r_0\ \sigma_0^2 + 105\ \sigma_0^4 + 10\ r_0^2\ (1 + 9\ \alpha\ \sigma_0^2))}{r_0^9}$$

$$C_6 = -\frac{\mu^3\ \sigma_0\ (-504\,\alpha\, r_0^3 + 225\,\alpha^2\ r_0^4 - 1260\ r_0\ \sigma_0^2 + 945\,\sigma_0^4 + 70\ r_0^2\ (4 + 15\,\alpha\,\sigma_0^2))}{r_0^{11}}$$

$$C_7 = \frac{1}{r_0^{13}}\ (\mu^{7/2}\ (-729\,\alpha^2\ r_0^5 + 225\,\alpha^3\ r_0^6 - 17325\ r_0\ \sigma_0^4 + 10395\,\sigma_0^6 +$$
$$1575\ r_0^2\ \sigma_0^2\ (4 + 9\,\alpha\,\sigma_0^2) + 7\,\alpha\ r_0^4\ (112 + 675\,\alpha\,\sigma_0^2) - 14\ r_0^3\ (20 + 783\,\alpha\,\sigma_0^2)))$$

$$C_8 = -\frac{1}{r_0^{15}}\ (\mu^4\ \sigma_0\ (-37206\,\alpha^2\ r_0^5 + 11025\,\alpha^3\ r_0^6 - 270270\ r_0\ \sigma_0^4 + 135135\,\sigma_0^6 +$$
$$3465\ r_0^2\ \sigma_0^2\ (40 + 63\,\alpha\,\sigma_0^2) + 945\,\alpha\ r_0^4\ (44 + 105\,\alpha\,\sigma_0^2) - 1540\ r_0^3\ (10 + 153\,\alpha\,\sigma_0^2)))$$

$$C_9 = -\frac{1}{r_0^{17}}\ (\mu^{9/2}\ (-48231\,\alpha^3\ r_0^7 + 11025\,\alpha^4\ r_0^8 - 4729725\ r_0\ \sigma_0^6 + 2027025\,\sigma_0^6 +$$
$$630630\ r_0^2\ \sigma_0^4\ (5 + 6\,\alpha\,\sigma_0^2) - 15015\ r_0^3\ \sigma_0^2\ (40 + 351\,\alpha\,\sigma_0^2) + 54\,\alpha^2 r_0^6 (1459 + 7350\,\alpha\,\sigma_0^2) -$$
$$55\,\alpha\ r_0^5 (1036 + 25029\,\alpha\,\sigma_0^2) + 770\ r_0^4 (20 + 2052\,\alpha\,\sigma_0^2 + 2835\,\alpha^2\ \sigma_0^4)))$$

$$C_{10} = -\frac{1}{r_0^{19}}\ (\mu^5\ \sigma_0\ (-4029300\,\alpha^3\ r_0^7 + 893025\,\alpha^4\ r_0^8 - 91891800\ r_0\,\sigma_0^6 +$$
$$34459425\,\sigma_0^8 + 2702700\ r_0^2\ \sigma_0^4\ (28 + 27\,\alpha\,\sigma_0^2) - 420420\ r_0^3\ \sigma_0^2\ (50 + 297\,\alpha\,\sigma_0^2) -$$
$$102960\,\alpha\ r_0^5\ (49 + 450\,\alpha\,\sigma_0^2) + 1188\,\alpha^2\ r_0^6\ (5707 + 11025\,\alpha\,\sigma_0^2) +$$
$$70070\ r_0^4 (20 + 774\,\alpha\,\sigma_0^2 + 729\,\alpha^2\ \sigma_0^4)))$$

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متسلسلة التعاكس لمعادلة كبلر الشاملة

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