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TRANSIENT GROUNDWATER FLOW IN HETEROGENEOUS GEOLOGICAL FORMATIONS

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Transient Groundwater Flow in Heterogeneous Geological Formations

السريان الانتقالى للمياه الجوفية فى التكوينات الجيولوجيه غير المتجانسة

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ا لاصة:

هذا البحث عرض اكه عددية للسريان غ ا ستقر للمياه ا وفية طبقات متجانسة و أخرى غ متجانسة. توليد البناء غ ا تجانس من الأ اط ا يولوجيه باستخدام وذج سلسلة ماركوف ا زدوجه. حل ا عادلات ا اكمه للسريان باستخدام طريقة الفروق ا دوده للسريان غ ا ستقر ا ياه ا وفية ا ستوى الأفقى و صائص غ متجانسه للطبقات.

أوضحت النتائج أن التغ يه مقدار الا دار ا يدروليكى العام مع عدم التجانس الطبقات يؤدى ا تغ يه مقدار و ا اه الا دار ا يدروليكى ا لى. كما أوضحت النتائج أن زيادة معامل التخزين يؤدى ا تنعيم رد فعل الطبقة ا امله للمياه صورة الضاغط ا يدروليكى و سرعة دارسى. و هذا التنعيم أكثر حالة الضاغط ا يدروليكى عنه سرعة دارسى.

أظهرت النتائج أيضا أنه حالة الطبقات غ ا تجانسة و ا درست هذا البحث أن الضاغط ا يدروليكى و سرعة دارسى ا انبيه كانت طور واحد بينما سرعة دارسى الا اه الطو كانت خارج الطور.

ABSTRACT

Numerical simulations of unsteady groundwater flow in homogeneous and artificially generated heterogeneous geological formations have been presented. The heterogeneous structure of the geological patterns has been generated using the coupled chain Markov model developed by Elfeki and Dekking [2001]. Solution of the governing equations is achieved through the application of a finite difference approach to the partial differential equation of unsteady groundwater flow in horizontal plane with heterogeneous properties. It has been shown that global gradient (regional gradient) magnitude variability coupled with aquifer heterogeneity generates local directional and magnitude gradient variabilities. Increasing of the storage coefficient leads to smoothing of the aquifer response in terms of hydraulic head and Darcy's velocity. However, the smoothing effect is more pronounced in the hydraulic heads when compared with Darcy's velocity. In heterogeneous aquifer presented in this study, the aquifer response in terms of hydraulic head field and the lateral Darcy's velocity are in phase with the input time series, however the longitudinal Darcy's velocity is out of phase.

1. INTRODUCTION

Transient flow conditions have strong influence on contaminant spreading in aquifers.

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This behavior has been supported by many field observations [e.g. Gelhar, 1993]. Significant progress of steady groundwater flows in stationary Gaussian and non-Gaussian random fields have been achieved [e.g. Smith and Freeze, 1979 and Ababou et al., 1989]. Many researchers show still vivid interest to describe the hydrodynamics of flow in heterogeneous fields under transient conditions. Only a limited number of studies are devoted to this area. Just recently, in the hydrogeological community, a considerable attention is made on evaluating the effects of transient conditions in heterogeneous media.

Two main transient conditions are causing the spreading: the gradient magnitude variability and the gradient direction variability. In the current research, a focus is made on the influence of gradient magnitude variability, which can be described as a multiple scale time series, combined with aquifer heterogeneity on flow characteristics.

The goal of this research is to investigate the hydrodynamics of flow under transient conditions in heterogeneous aquifer. Unsteady groundwater flow model in a heterogeneous confined aquifer has been developed. The model is based on a finite difference numerical scheme in terms of potentials. The model is used to study the influence of transient conditions (gradient magnitude variability) on groundwater flow behavior at multiple time scales. In this model, the influence of water level fluctuations in a river, that is feeding an aquifer (see Figure 1), on the hydraulic head and local Darcy's velocity fluctuations are considered. Simulations have been performed by the developed model under two different input signals in homogeneous and heterogeneous media with a large scale spatial variability. The first signal is a sudden drop in the water level in the river side and the second is a time series in a form of two components a random signal superimposed over a cosine wave.

2. GOVERNING EQUATIONS OF UNSTEADY GROUNDWATER FLOW PROBLEM

The governing equation, in the absence of source and sink terms, of unsteady two-dimensional (in horizontal plane) saturated incompressible fluid flow in an anisotropic heterogeneous confined aquifer is given by,

$$S\frac{\partial \Phi(x,y,t)}{\partial t} = \frac{\partial}{\partial x}\left(T_{xx}(x,y)\frac{\partial \Phi(x,y,t)}{\partial x}\right) + \frac{\partial}{\partial y}\left(T_{yy}(x,y)\frac{\partial \Phi(x,y,t)}{\partial y}\right) \qquad \in \Omega \tag{1}$$

where T_{xx} (x,y) is the transmisivity in x-direction, T_{yy} (x,y) is transimisivity in y-direction, $\Phi(x,y,t)$ is hydraulic or piezometric head, S is the storage coefficient, and Ω is domain of interest.
The transmisivity is related to the hydraulic conductivity by,

$$\begin{aligned} T_{xx}(x,y) &= K_{xx}(x,y)H(x,y) \\ T_{yy}(x,y) &= K_{yy}(x,y)H(x,y) \end{aligned} \tag{2}$$

where $H(x,y)$ is the aquifer thickness at location x and y.

No-flow (Neumann condition) or constant head (Dirichlet condition) are specified on the boundaries of the flow domain, that is,

$$\frac{\partial}{\partial n}\left[\Phi(x,y,t)\right]=0 \qquad \text{on } x,y\in\Gamma_1$$
$$\Phi(x,y,t)=\Phi_o \qquad \text{on } x,y\in\Gamma_2 \qquad (3)$$
$$\Phi(x,y,t)=\Phi(t) \qquad \text{on } x,y\in\Gamma_3$$

where Γ is boundary of the domain, $\Gamma_1 + \Gamma_2 + \Gamma_3 = \Gamma$, n is the unit vector normal to the boundary pointing outward, and Φ_o is the prescribed head.

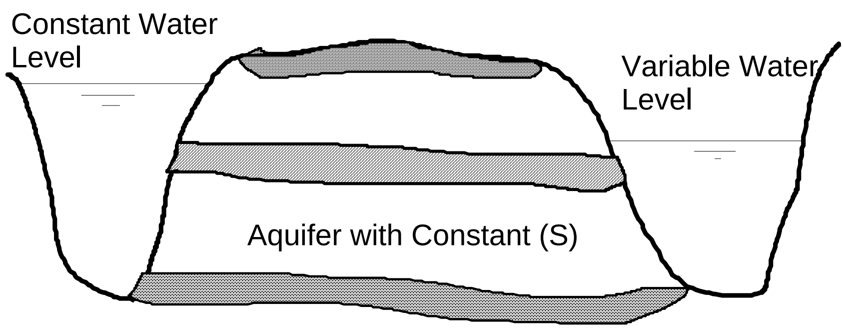

Figure 1. Transient Groundwater Flow in Confined Aquifer.

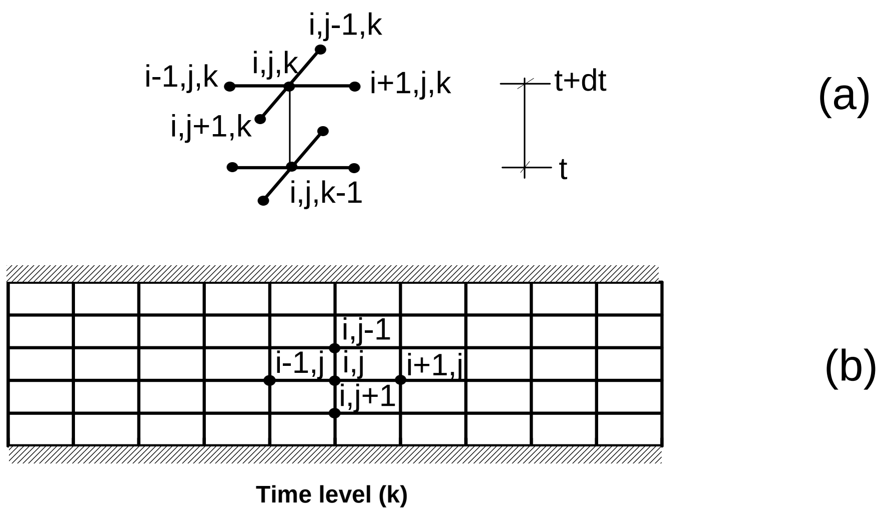

Figure 2. Domain Descretization (b) and The Fully Implicit Numerical Scheme (a).

3. FINITE DIFFERENCE FORMULATION AND SOLUTION BY CONJUGATE GRADIENT METHOD

A finite difference model has been developed for discretization of Eq.(1). A numerical scheme with a five-points operator shown in Figure 2 is used. The finite difference analog for the derivatives are given in the following expressions,

$$T_{xx}(x,y)\left(\frac{\partial\Phi(x,y,t)}{\partial x}\right)\approx T_{xx_{i+1/2,j}}\left[\frac{\Phi^k_{i+1,j}-\Phi^k_{i,j}}{\Delta x}\right] \tag{4}$$

where $T_{xx_{i+1/2,j}}$ is the interface transmisivity between node (i+1,j) and node (i,j). This transmisivity could be estimated by the harmonic mean of the surrounding nodes in x-direction,

$$T_{xx_{i+1/2,j}}=\frac{2T_{xx_{i+1,j}}T_{xx_{i,j}}}{T_{xx_{i+1,j}}+T_{xx_{i,j}}} \tag{5}$$

and $\Phi^k_{i,j}$ is the hydraulic head at node (i,j) at time k.

Similarly,

$$T_{yy}(x,y)\left(\frac{\partial\Phi(x,y,t)}{\partial y}\right)\approx T_{yy_{i,j+1/2}}\left[\frac{\Phi^k_{i,j+1}-\Phi^k_{i,j}}{\Delta y}\right] \tag{6}$$

with $T_{yy_{i,j+1/2}}$ is given by

$$T_{yy_{i,j+1/2}}=\frac{2T_{yy_{i,j+1}}T_{yy_{i,j}}}{T_{yy_{i,j+1}}+T_{yy_{i,j}}} \tag{7}$$

Further evaluation leads to

$$\frac{\partial}{\partial x}\left(T_{xx}(x,y)\frac{\partial\Phi(x,y,t)}{\partial x}\right)\approx\frac{T_{xx_{i+1/2,j}}\left[\dfrac{\Phi^k_{i+1,j}-\Phi^k_{i,j}}{\Delta x}\right]-T_{xx_{i-1/2,j}}\left[\dfrac{\Phi^k_{i,j}-\Phi^k_{i-1,j}}{\Delta x}\right]}{\Delta x} \tag{8}$$

$$\frac{\partial}{\partial y}\left(T_{yy}(x,y)\frac{\partial\Phi(x,y,t)}{\partial y}\right)\approx\frac{T_{yy_{i,j+1/2}}\left[\dfrac{\Phi^k_{i,j+1}-\Phi^k_{i,j}}{\Delta y}\right]-T_{yy_{i,j-1/2}}\left[\dfrac{\Phi^k_{i,j}-\Phi^k_{i,j-1}}{\Delta y}\right]}{\Delta y} \tag{9}$$

$$S\frac{\partial\Phi(x,y,t)}{\partial t}\approx S_{i,j}\frac{\left[\Phi^k_{i,j}-\Phi^{k-1}_{i,j}\right]}{\Delta t} \tag{10}$$

substitution of Eq.(8), Eq.(9) and Eq.(10) into Eq.(1) leads to the finite difference analog for the partial differential equation as,

$$A_{i,j}\Phi^{k}_{i+1,j} + B_{i,j}\Phi^{k}_{i,j-1} + C_{i,j}\Phi^{k}_{i-1,j} + D_{i,j}\Phi^{k}_{i,j+1} + E_{i,j}\Phi^{k-1}_{i,j} - F_{i,j}\Phi^{k}_{i+1,j} = 0$$

where,

$$\begin{aligned} A_{i,j} &= T_{xx_{i+1/2,j}} / \Delta x^2 \\ B_{i,j} &= T_{yy_{i,j+1/2}} / \Delta y^2 \\ C_{i,j} &= T_{yy_{i,j-1/2}} / \Delta x^2 \\ D_{i,j} &= T_{yy_{i,j-1/2}} / \Delta y^2 \\ E_{i,j} &= S_{i,j} / \Delta t \\ F_{i,j} &= A_{i,j} + B_{i,j} + C_{i,j} + D_{i,j} + E_{i,j} \end{aligned} \tag{11}$$

After the solution of the flow equation, one can calculate the potential head distribution at each time step and consequently the gradient field and the Darcy's velocity field on the grid. This is can be done by differentiation as,

$$q^{k}_{x_{i+1/2,j}} = -T_{xx}(x,y)\left(\frac{\partial \Phi(x,y,t)}{\partial x}\right) \approx -T_{xx_{i+1/2,j}}\left[\frac{\Phi^{k}_{i+1,j} - \Phi^{k}_{i,j}}{\Delta x}\right] \tag{12}$$

$$q^{k}_{y_{i,j+1/2}} = -T_{yy}(x,y)\left(\frac{\partial \Phi(x,y,t)}{\partial y}\right) \approx -T_{yy_{i,j+1/2}}\left[\frac{\Phi^{k}_{i,j+1} - \Phi^{k}_{i,j}}{\Delta y}\right] \tag{13}$$

where $q^{k}_{x_{i+1/2,j}}$ and $q^{k}{}_{yi,j+1}$ are the inter nodal Darcy's velocity components between nodes (i,j) and ($i+1,j$), and between nodes (i,j) and ($i,j+1$) at time k.

From the Darcy's velocities the pore-velocities are calculated by dividing Eqs.(12) and (13) by the effective porosity of the medium. This is essential to transport models that will be considered in the future.

A large number of solvers are available for systems of linear equations and some of the efficient solvers, in case of heterogeneous systems with large number of nodes, are the iterative ones. All the iterative solvers start with an initial guess of the field variable and in each iteration a new and better approximation is computed. It has been proven that the method of conjugate gradient (CG) is powerful in addressing highly heterogeneous medium. This method is adopted by Elfeki [1996] for steady state flow problems. The CG method is extended in the current study to handle time dependent flow problems. The formulas and the algorithm for implementation in case of transient conditions are presented. The algorithm used here is an extension of the one given by Strikwerda [1989]. Some modifications are adopted to handle the heterogeneity of the medium and transient conditions. A backward difference fully implicit scheme solved by CG is used for the time integration. This technique is fairly simple, completely stable and is free from oscillation problems. The equations to solve are in the form of Eq.(11) which form a linear system $\boldsymbol{ax} = \boldsymbol{b}$ where, $\boldsymbol{a}$ is positive definite matrix and the vector $\boldsymbol{b}$ contains both zeros and the values of the solution on the boundary. The procedure involves the following steps between two successive time steps k and $k+1$.

First step: an initial iterate $\Phi^{k(0)}{}_{i,j}$ is given and then the residual $r^{k(0)}{}_{i,j}$ is computed as,

$$r_{i,j}^{k(0)} = A_{i,j}\Phi_{i+1,j}^{k(0)} + B_{i,j}\Phi_{i,j-1}^{k(0)} + C_{i,j}\Phi_{i-1,j}^{k(0)} + D_{i,j}\Phi_{i,j+1}^{k(0)} + E_{i,j}\Phi_{i,j}^{k-1(0)} - F_{i,j}\Phi_{i,j}^{k(0)} \tag{14}$$

A matrix $P^{k(0)}{}_{i,j}$ is introduced as

$$P_{i,j}^{k(0)} = r_{i,j}^{k(0)} \tag{15}$$

with $|r^{k(0)}|^2$ also being computed by accumulating the products ($r_{i,j}{}^{k(0)}.r_{i,j}{}^{k(0)}$). In a mathematical form is given by,

$$|r^{k(0)}|^2 = \sum_i \sum_j \left[r_{i,j}^{k(0)}\right]^2 \tag{16}$$

Another matrix $Q^{k(0)}{}_{i,j}$ is introduced and computed as

$$Q_{i,j}^{k(0)} = -A_{i,j}r_{i+1,j}^{k(0)} - B_{i,j}r_{i,j-1}^{k(0)} - C_{i,j}r_{i-1,j}^{k(0)} - D_{i,j}r_{i,j+1}^{k(0)} + E_{i,j}r_{i,j}^{k(0)} \tag{17}$$

and the inner product ($P^{k(0)}, Q^{k(0)}$), is computed by accumulating the product $P_{i,j}{}^{k(0)}$ $Q_{i,j}{}^{k(0)}$ to evaluate the parameter $\alpha^{k(0)}$ as,

$$\alpha^{k(0)} = \frac{|r^{k(0)}|^2}{(P^{k(0)}, Q^{k(0)})} \tag{18}$$

Note that for Dirichlet boundary condition (prescribed head boundary) $r^{k(m)}$, $P^{k(m)}$, and $Q^{k(m)}$ where m denotes the iteration number, should be zero on the boundary.

Second step: begin the main computation loop. $\Phi_{i,j}{}^{k(m)}$ and $r_{i,j}{}^{k(m)}$ are updated by

$$\begin{aligned} \Phi_{i,j}^{k(m+1)} &= \Phi_{i,j}^{k(m)} + \alpha^{k(m)} P_{i,j}^{k(m)} \\ r_{i,j}^{k(m+1)} &= r_{i,j}^{k(m)} - \alpha^{k(m)} Q_{i,j}^{k(m)} \end{aligned} \tag{19}$$

with $|r^{k(m+1)}|^2$ also computed. Another parameter $\beta^{k(m+1)}$ is computed by the formula,

$$\beta^{k(m)} = \frac{|r^{k(m+1)}|^2}{|r^{k(m)}|^2} \tag{20}$$

then P and Q are updated by

$$\begin{gathered} P_{i,j}^{k(m+1)} = r_{i,j}^{k(m+1)} + \beta_k P_{i,j}^{k(m)} \\ Q_{i,j}^{k(m+1)} = \left[E_{i,j}\, r_{i,j}^{k(m+1)} - A_{i,j}\, r_{i+1,j}^{k(m+1)} - B_{i,j}\, r_{i,j-1}^{k(m+1)} - C_{i,j}\, r_{i-1,j}^{k(m+1)} - D_{i,j}\, r_{i,j+1}^{k(m+1)}\right] + \beta_k Q_{i,j}^{k(m)} \end{gathered} \tag{21}$$

and the inner product ($P^{k(m+1)}$, $Q^{k(m+1)}$) is computed.

Third step: $\alpha^{k(m+1)}$ is computed as the ratio,

$$\alpha^{k(m+1)} = \frac{|r^{k(m+1)}|^2}{(P^{k(m+1)}, Q^{k(m+1)})} \tag{22}$$

and m is incremented.

The conjugate gradient method is terminated when $|r^{k(m)}|$ is sufficiently small. As with the general iterative methods, the method should be continued until the error in the iteration is comparable to the truncation error in the numerical scheme. Table 1 displays the numerical values used to perform the simulations in homogeneous and heterogeneous cases under sudden drop in water level and time series boundary conditions.

Table 1. Simulation Parameters used in Computation

Parameter	Numerical Value
Domain dimensions	200m × 50m
Domain discretization	1.0m × 1.0m
Time step	0.5 day
Upstream Fixed Head Boundary	20 m
Downstream Sudden drop Head Boundary	10 m
Constant Aquifer Thickness	10 m
Homogeneous Hydraulic Conductivity	10 m/day
Heterogeneous Hydraulic Conductivity	1.0, 10., 50., 100. m/day
Accuracy in Computation	0.001
No. of Time Steps	50 Steps (25 days)
Storage Coefficient	0.00001, 0.0001, 0.001, 0.01, 0.1

4. ANALYSIS OF MODEL RESULTS

In the homogeneous aquifer presented in this simulation (Figure 3), it is found that increasing the storage coefficient leads to smoothing and delaying in the aquifer response in terms of hydraulic head and Darcy's velocity. The aquifer response in terms of hydraulic head is in phase with the input time series however, the longitudinal Darcy's velocity is out of phase.

In a heterogeneous aquifer, a geological structure with realistic characteristics is generated and displayed in Figure 4. Figure 5 shows the simulation under sudden drop in water level at the right boundary (S = 0.01). The Figure shows the propagation of the groundwater head over a record of 9 days when steady state condition is almost achieved. The result in case of time series boundary is displayed in Figure 6. The simulations show different responses according to the value of the storage coefficient. Similar to the homogeneous case, increasing of the storage coefficient leads to smoothing and delaying in the aquifer response in terms of hydraulic head and Darcy's velocity. However, the smoothing effect is more pronounced in the hydraulic head when compared with Darcy's velocity. The aquifer response in terms of hydraulic head field and the lateral Darcy's velocity are in phase with the input time series, however the longitudinal Darcy's velocity is out of phase. Global gradient (regional gradient) magnitude variability coupled with aquifer heterogeneity generates local directional and magnitude gradient variabilities.

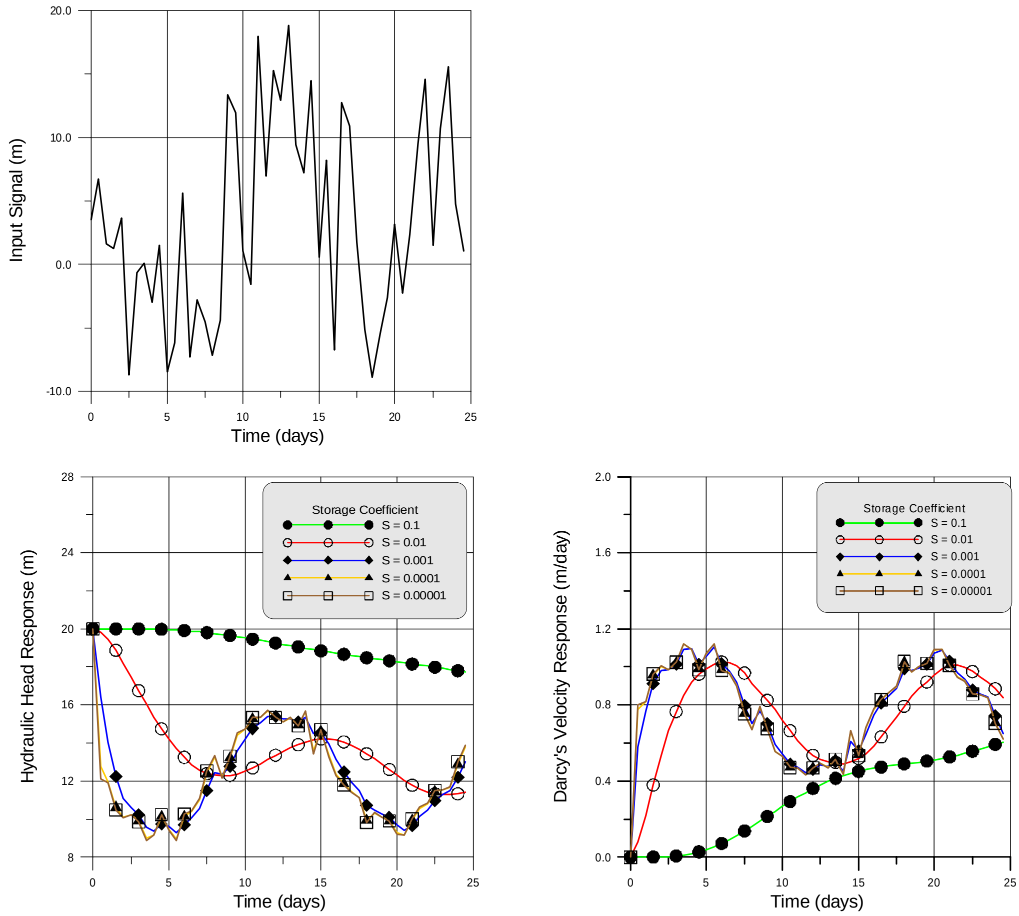

Figure 3. Numerical Simulation in Case of Homogenous Medium under Time Series Boundary at the Water Level (Top left most graph is the input signal, bottom left graph is the aquifer response in terms of hydraulic head at the middle of the aquifer and the bottom right graph is the aquifer response in terms of Darcy's velocity at the same location.

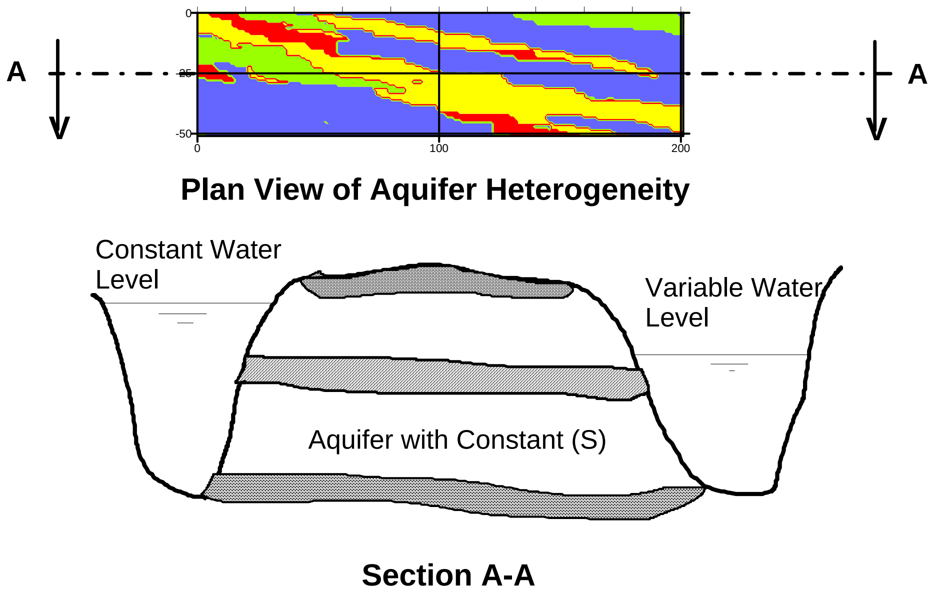

Figure 4. Transient Groundwater Flow in Heterogeneous Confined Aquifer.

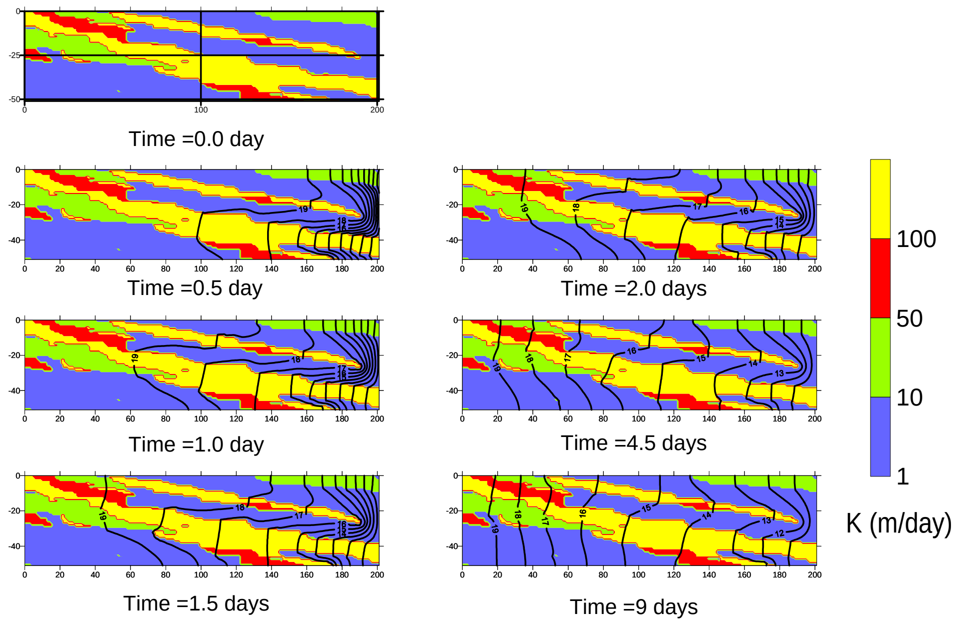

Figure 5. Snapshots of The Hydraulic Head Distribution under Sudden Drop in Water Level in Heterogeneous Confined Aquifer (Steady state condition is almost reached after 9 days).

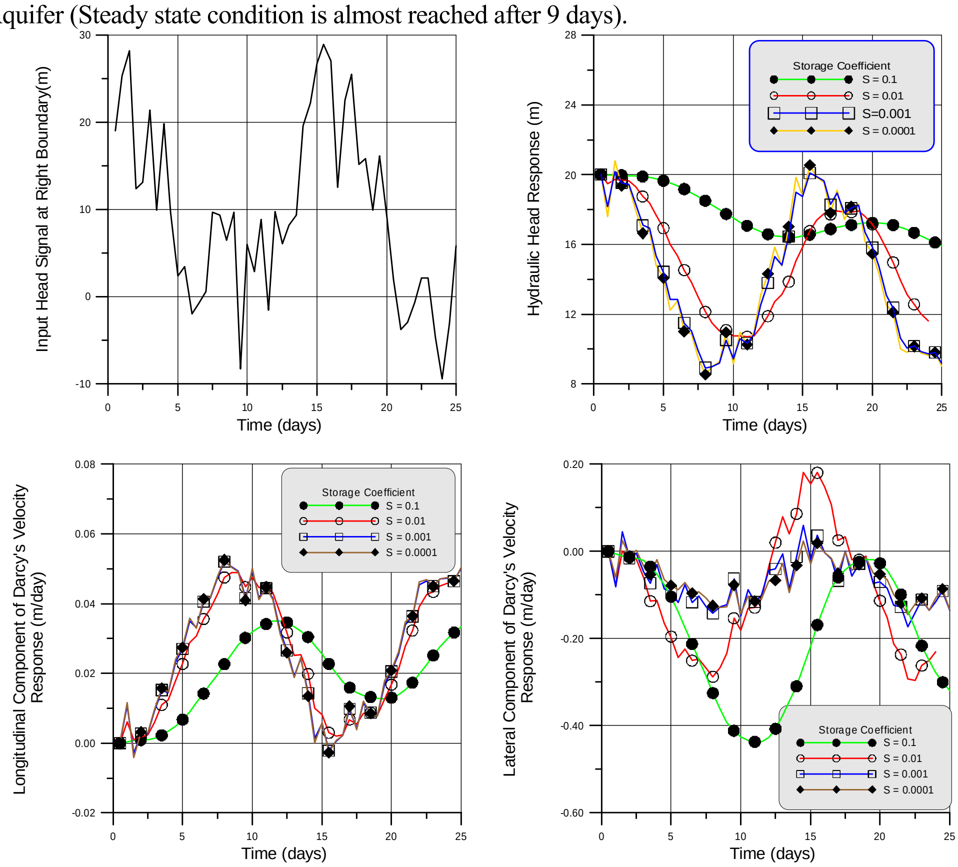

Figure 6. Numerical Simulation Results in Case of Heterogeneous Medium under Time Series Boundary at the Water Level. Top left most graph is the input signal, right most graph is the aquifer response in terms of hydraulic head at the middle of the aquifer, the bottom left graph is the aquifer response in terms of longitudinal Darcy's velocity at the same location and the bottom right graph is the aquifer response in terms of lateral Darcy's velocity at

the same location.

5. CONCLUSIONS

Numerical simulations of groundwater flow in homogeneous and artificially generated heterogeneous geological formations have been presented. The heterogeneous structure of the geological patterns has been generated using the coupled chain Markov model developed by Elfeki and Dekking [2001]. Solution of the governing equations is achieved through the application of a finite difference approach to the partial differential equation of unsteady groundwater flow in horizontal plane with heterogeneous properties. The solution algorithm is an extension of the CG method used by Elfeki [1996] to handle time dependent problems. The following conclusions can be drawn from this research:

1. Global gradient (regional gradient) magnitude variability coupled with aquifer heterogeneity generates local directional and magnitude gradient variabilities.

2. Increasing the storage coefficient leads to smoothing and delaying in the aquifer response in terms of hydraulic head and Darcy's velocity. The smoothing effect is more pronounced in the hydraulic heads when compared with Darcy's velocity.

3. In the homogeneous aquifer presented in this simulation, the aquifer response in terms of hydraulic head field is in phase with the input time series however, the longitudinal Darcy's velocity is out of phase.

4. In the heterogeneous aquifer presented in this study, the aquifer response in terms of hydraulic head field and the lateral Darcy's velocity are in phase with the input time series, however the longitudinal Darcy's velocity is out of phase.

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